IE410 Qualifier Exam
Fall 2015

1. [20 points] Let's think through how exponential random variables combine. Let τ_1 and τ_2 be independent exponential random variables with parameters, respectively, λ_1 and λ_2. Define

$$\tau \stackrel{\text{def}}{=} \min\{\tau_1, \tau_2\}.$$

   (a) [5 points] Compute $\mathbb{P}\{\tau > t,\ \tau_1 < \tau_2\}$.

   (b) [5 points] Compute $\mathbb{P}\{\tau_1 = \tau_2\}$.

   (c) [5 points] Compute $\mathbb{P}\{\tau > t\}$ (easy from Question a and symmetry).

   (d) [5 points] Compute $\mathbb{P}\{\tau_1 < \tau_2\}$ (easy from Question a).

   (e) [5 points] Prove that τ and $\{\tau_1 < \tau_2\}$ are independent.

   This, of course, is the calculation which allows one to decompose a continuous-time Markov chain into exponential times and an embedded Markov process.

2. [15 points] Suppose that X is a continuous-time Markov chain with transition rates

$$\mathbb{P}\{X_{t+\delta} = j | X_t = i\} \approx q_{i,j}\delta$$

   for $j \neq i$, where

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -7 & 4 & 0 & 3 \\ 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & -6 & 6 \\ 0 & 0 & 2 & 1 & -3 \end{pmatrix}$$

   The diagonal of Q is defined via standard procedures.

   (a) [5 points] What are the communicating classes?

   (b) [5 points] What are the transient states?

   (c) [5 points] Let $\hat{X}$ be the embedded Markov chain; set

$$\begin{aligned} \tau_0 &\stackrel{\text{def}}{=} 0 \\ \tau_{n+1} &\stackrel{\text{def}}{=} \inf\{t > \tau_n : X_t \neq X_{\tau_n}\} \\ \hat{X}_n &\stackrel{\text{def}}{=} X_{\tau_n}. \end{aligned}$$

   Compute the state transition matrix P of $\hat{X}$.

   (d) Compute $\mathbb{E}[\tau_1 | X_0 = 2]$.

3. [10 points] Let's wait, but only a finite time. Let X be exponential(1) and define

$$Y \stackrel{\text{def}}{=} \min\{X, 1\}.$$

   (a) [5 points] Compute $\mathbb{P}\{Y = 1\}$

   (b) [5 points] Compute the cdf F_Y of Y (you might try, just for practice, first computing $F_Y(0.3)$ and $F_Y(4)$).