1. **20 points** Let’s think through how exponential random variables combine. Let $\tau_1$ and $\tau_2$ be independent exponential random variables with parameters, respectively, $\lambda_1$ and $\lambda_2$. Define

$$\tau \overset{\text{def}}{=} \min\{\tau_1, \tau_2\}.$$

(a) **5 points** Compute $\mathbb{P}\{\tau > t, \tau_1 < \tau_2\}$.

(b) **5 points** Compute $\mathbb{P}\{\tau_1 = \tau_2\}$.

(c) **5 points** Compute $\mathbb{P}\{\tau > t\}$ (easy from Question a and symmetry).

(d) **5 points** Compute $\mathbb{P}\{\tau_1 < \tau_2\}$ (easy from Question a).

(e) **5 points** Prove that $\tau$ and $\{\tau_1 < \tau_2\}$ are independent.

This, of course, is the calculation which allows one to decompose a continuous-time Markov chain into exponential times and an embedded Markov process.

2. **15 points** Suppose that $X$ is a continuous-time Markov chain with transition rates

$$\mathbb{P}\{X_{t+\delta} = j | X_t = i\} \approx q_{i,j} \delta$$

for $j \neq i$, where

$$Q = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
0 & -7 & 4 & 0 & 3 \\
0 & 0 & -5 & 5 & 0 \\
0 & 0 & 0 & -6 & 6 \\
0 & 0 & 2 & 1 & -3
\end{pmatrix}$$

The diagonal of $Q$ is defined via standard procedures.

(a) **5 points** What are the communicating classes?

(b) **5 points** What are the transient states?

(c) **5 points** Let $\hat{X}$ be the embedded Markov chain; set

$$\tau_0 \overset{\text{def}}{=} 0$$

$$\tau_{n+1} \overset{\text{def}}{=} \inf\{t > \tau_n : X_t \neq X_{\tau_n}\}$$

$$\hat{X}_n \overset{\text{def}}{=} X_{\tau_n}.$$  

Compute the state transition matrix $P$ of $\hat{X}$.

(d) Compute $\mathbb{E}[\tau_1 | X_0 = 2]$.

3. **10 points** Let’s wait, but only a finite time. Let $X$ be exponential(1) and define

$$Y \overset{\text{def}}{=} \min\{X, 1\}.$$

(a) **5 points** Compute $\mathbb{P}\{Y = 1\}$

(b) **5 points** Compute the cdf $F_Y$ of $Y$ (you might try, just for practice, first computing $F_Y(0.3)$ and $F_Y(4)$).